RAIRO-Oper. Res. 58 (2024) 1375–1399
https://doi.org/10.1051/ro/2024039

RAIRO Operations Research
www.rairo-ro.org

BID-ASK SPREAD DYNAMICS: LARGE UPWARD JUMP WITH GEOMETRIC CATASTROPHES

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Abstract. We propose a simple continuous-time stochastic model for capturing the dynamics of a limit order book in the presence of liquidity fluctuations, manifested by gaps in filled price levels within the OB. Inspired by [D. Farmer, L. Gillemot, F. Lillo, S. Mike and A. Sen, *Quant. Finance* **4** (2004) 383–397.], we define a model for the dynamics of spread that incorporates liquidity fluctuations and undertake a comprehensive theoretical study of the model's properties, providing rigorous proofs of several key asymptotic theorems. Furthermore, we show how large deviations manifest in the spread under this regime.

Mathematics Subject Classification. 60J27, 60F05, 60F10.

Received September 19, 2023. Accepted February 10, 2024.

1. Introduction

For several decades, one of the most challenging problems in quantitative finance has been understanding the underlying causes of asset price fluctuations in financial markets. While this problem has not yet been completely resolved, several studies have provided partial answers to the possible causes of these fluctuations, and some conclusion is that the evolution of prices in financial markets results from the interaction of buy and sell orders through a rather complex dynamic process. When this interaction is broken or disrupted, it can lead to significant price fluctuations and volatility in the market, see for instance [1, 6, 12, 13, 15–17, 33] and [7].

Another possible cause widely used by practitioners is that price fluctuations are essentially caused by the volume of orders. For example, Clark [8] and Gabaix *et al.* [20] have suggested this. On the other hand, Farmer *et al.* [17] showed that price fluctuations caused by individual market orders are essentially independent of the volume of orders. They found that large price fluctuations are mainly driven by liquidity fluctuations, which are variations in the market's ability to absorb new orders. The stochastic model for the dynamics of a limit Order Book (OB) when there are liquidity fluctuations proposed in this work was inspired by this last article. In

Keywords. Markov models, limit order book, geometric catastrophes, liquidity fluctuations.

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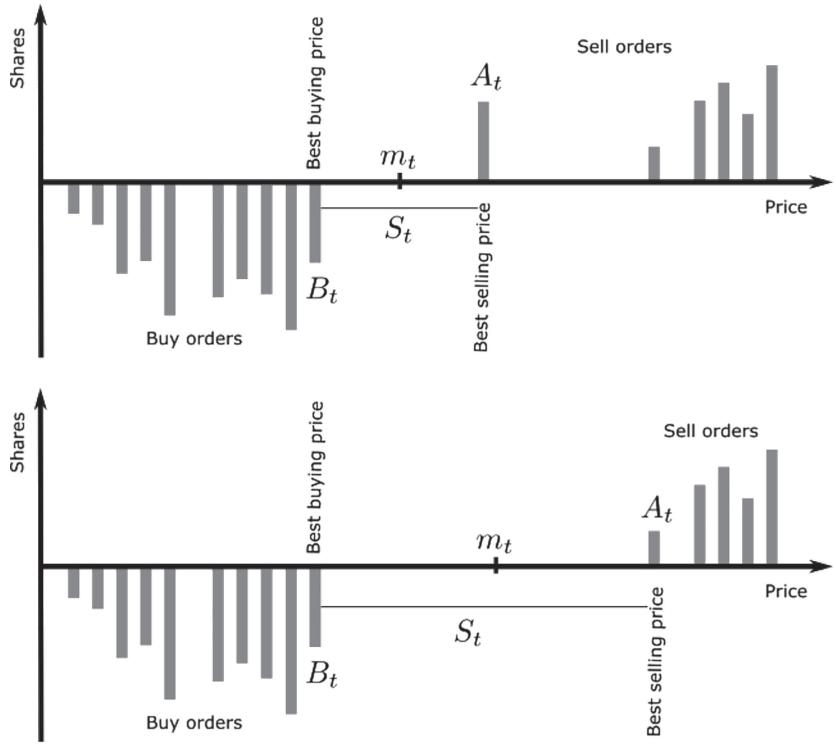

Figure 1. This figure shows the volume of limit orders at each price level. Sell limit orders should be displayed on the right side, while buy limit orders should be displayed on the left side. Note that when a market order to buy is executed, all the volume at the best ask is removed, resulting in a new configuration of the limit order book and a substantial change in the midpoint price.

this study, we propose a model designed to address temporary liquidity crises leading to gaps within the Order Book (OB). It is important to emphasize that our model specifically concentrates on a subset of liquidity crises and does not encompass various other regimes discussed in the literature. For a comprehensive understanding of diverse liquidity crisis scenarios, readers are encouraged to explore works such as [6,19,23,36], and [39]. Our approach is inspired by the concepts outlined in [17] for modeling the spread when significant price changes result from discrete fluctuations in liquidity, manifested by gaps in filled price levels within the OB. In Section 1, we illustrated a typical configuration of the OB before and after a large price fluctuation.

Farmer *et al.* [17] propose that large price changes are caused by gaps in the OB. When such a gap exists next to the best price, a new order can remove the best quote, triggering a large midpoint price change. Thus, the distribution of large price changes merely reflects the distribution of gaps in the OB. This view is supported by empirical evidence showing that the distribution of large price changes has a power-law tail, which is consistent with the distribution of gaps in the limit order book [7]. Moreover, the authors argue that the dynamics of order book gaps are driven by the arrival of new information and the resulting update of market participants' beliefs, rather than by the trading behavior of individual investors. This suggests that large price changes are more likely to occur when there is a sudden change in the market's perception of the value of an asset, rather than due to the actions of individual traders. See [16,17] and [7] for more details.

In general, the order book serves as an electronic register encompassing bid and ask prices alongside their corresponding quantities, see Figure 1. The availability of order book data is crucial for examining the mechanisms of price formation and fluctuations. The evolution of the order book arises from the intricate interplay between buy and sell orders, encompassing a complex dynamic process. Consequently, it has garnered significant attention and extensive exploration within the realms of market microstructure and econophysics literature. See for instance [5, 7, 15, 37]. Based on empirical characteristics presented in these studies, several models for the evolution of the OB have been proposed more recently, including [3, 9, 12–14].

For a comprehensive literature review on order book dynamics, a good starting point would be the works of Farmer *et al.* [17], Smith *et al.* [37] and Bouchaud *et al.* [7], and their respective reference lists for further reading. Fully comprehending the order book (OB) in financial markets is an extremely complex task because there are numerous factors that influence it. Factors such as market sentiment, liquidity, trading activity, and news events can all have a significant impact on the order book and its dynamics.

By incorporating insights from Farmer *et al.* [17] and empirical evidence presented by Bouchaud *et al.* [7], we introduce a simple theoretical stochastic model for the order book and the spread in this study. According to the proposed model, the spread is determined as the disparity between two processes. The first process entails a Poisson process featuring upward jumps, with their magnitudes corresponding to the average gap in the order book resulting from liquidity shocks. The second process involves a Poisson flow of so-called *geometric catastrophes*, which systematically reduce the spread.

Our model sheds light on the relationship between the best bid and ask prices and the spread in this regime, as well as how changes in the order book can affect the spread dynamics. In Figure 1, we show how the presence of a gap in the order book adjacent to the best price can lead to a situation where a new order is able to remove the best quote, resulting in a substantial change in the midpoint price.

Our research aligns with the objective of comprehending and modeling the effects of liquidity shocks on large price changes. We propose and investigate a simple continuous-time Markov process that characterizes the dynamics of the order book. We established stability properties, the law of large numbers, and the central limit theorem. We also analyze how large deviations in the spread occur employing the methodologies presented in [25, 28, 29].

The rest of the paper is organized as follows. In Section 2, we provide the motivation and description of our model for the Order Book and the spread. Section 3 is dedicated to discussing the stability conditions for the Markov chain that characterizes the dynamics of the spread. We also establish conditions for the existence of an invariant measure, given by $\gamma_+ G - \frac{\gamma_-}{p} < 0$. In Section 4, we employ the forward Kolmogorov equation to calculate the invariant measure of the spread process. Section 5 is dedicated to establishing the law of large numbers and the central limit theorem for the embedding Markov chain of the bid-spread process. Finally, in Section 6, we establish the local large deviation principle (LLDP) and conduct an analysis of how large deviations in the spread occur. This is followed with concluding remarks in Section 7. The Appendix contains the proofs of the stated results.

2. Motivation and description of the model

The mechanism of price formation in modern financial markets is commonly referred to as the continuous double auction and has been extensively studied in the literature. For more details on this topic, see [37]. In particular, probabilistic models based on Markov chains have been proposed to model the continuous double auction. Examples include [3, 9, 13, 14], and [12].

In the OB, agents can place different types of orders, which can be grouped into *market orders*, which are requests to buy or sell a given number of shares immediately at the best available price; *limit orders* or *quotes*, which also state a limit price P_t. Limit orders often do not result in an immediate transaction and are stored in a queue called the *limit order book*. Additionally, we have cancellations. Buy limit orders are called *bids*, and sell limit orders are called *asks*. At any given time, there is a best (lowest) offer to sell with a price of A_t and a best (highest) bid to buy with a price of B_t (see Fig. 1). The price gap between them is called the *spread*, denoted by $S_t = A_t - B_t$. Prices are not continuous but rather change in discrete quanta called *ticks*, with a size of δ. The number of shares in an order is called either its *size* or its *volume*. The *midpoint price* or *mid-price* is defined as follows:

$$m_t = \frac{A_t + B_t}{2}.$$

Then,

$$B_t = m_t - \frac{S_t}{2} \quad \text{and} \quad A_t = m_t + \frac{S_t}{2} \tag{1}$$

The above equation is important because it reflects what Farmer *et al.* [17,33], and [7] suggested: studying and analyzing the spread can provide information on price fluctuations. Note that the price changes are always in the same direction: a buy market order will either leave the best ask the same or make it bigger, and a sell market order will either leave the best bid the same or make it smaller. The result is that buy market orders can increase the *mid-price* m_t, and sell orders can decrease it. In general, to study the dynamics of the price of an asset, stochastic models are usually proposed for the mid-price m_t, the spread S_t, and the dynamics of the order book (see for instance [2,11,13]). In this work, in order to study the cause of large fluctuations in the price of a stock, in the presence of liquidity shocks, we use the spread of the asset.

The assumption that large price changes are caused by large market orders is very natural. Surprisingly, this is not the main cause of most large price changes. In [17] authors demonstrate that large price changes are due to discrete fluctuations in liquidity, manifested by *gaps* in filled price levels in the limit order book (see Fig. 1). For instance, if the best ask is removed, a huge return only indicates a sizable gap inside the Order Book (OB). In this work, we propose a simple Markov dynamic model for the spread, aiming to replicate empirical facts presented in [17] and [33]. Our objective is to derive theoretical results concerning the existence of an invariant measure within the model and to explore considerations related to large deviations.

Here we assume a tick size δ is equal to 1. We denote the mean gap in the order book, caused by liquidity shocks, as G, which is assumed to be a positive integer. Let B_t be the (best) bid price and A_t be the (best) ask price. In general, the temporal dynamics of prices can be described by a continuous-time process $X_t = (B_t, A_t)$, with values in the discrete state space $\mathbb{X} \in \mathbb{Z} \times \mathbb{Z}$,

$$\mathbb{X} = \{(b, a) \in \mathbb{Z} \times \mathbb{Z} : b < a\},$$

where $\mathbb{Z}$ stands for integers.

The transitions of the chain X_t are defined by the following transition rates: let $(b, a) \in \mathbb{X}$ be a state of the Markov chain, then

$$\begin{array}{l} (b,a) \to (b, a+\Delta) \text{ with rate } \alpha_+(\Delta), \\ (b,a) \to (b, a-\Delta) \text{ with rate } \alpha_-(\Delta), \text{ where } 0 < \Delta < a-b, \\ (b,a) \to (b-\Delta, a) \text{ with rate } \beta_-(\Delta), \\ (b,a) \to (b+\Delta, a) \text{ with rate } \beta_+(\Delta), \text{ where } 0 < \Delta < a-b. \end{array} \tag{2}$$

In all cases, the increments Δ are positive integers. The function $\alpha_+(\cdot)$ (resp. $\beta_-(\cdot)$) represents the rate at which the ask (resp. bid) price increases (resp. decreases) in Δ ticks due to the execution of market buy (resp. sell) orders or cancellations of limited sell (resp. buy) orders. On the other hand, the functions $\alpha_-(\cdot)$ (resp. $\beta_+(\cdot)$) represent the rate at which the ask (resp. bid) price decreases (resp. increases) due to a limited sell (resp. buy) order placed within the spread. In general, during temporary liquidity crises where spreads are large, there is intense competition for liquidity provision, leading to an increased probability of spreads closing and reverting to normal values (see *e.g.* [33] and [17]). Consequently, it is reasonable to assume that the rate of spread closing is contingent upon the spread's size, denoted as $\alpha_-(\Delta, b-a)$ and $\beta_+(\Delta, b-a)$ (see, for instance [19]). However, for the sake of simplicity, we omit this dependency for two primary reasons: to ensure the existence of an invariant measure and to facilitate considerations regarding large deviations (see Section 4 and 6).

In this work, we study the asymptotic behavior of X_t as t goes to infinity. To accomplish this, it will be easy to use the relation (1) and consider the equivalent Markov process $Y_t = (B_t, S_t)$ with state space $\mathbb{Z} \times \mathbb{N}$. Although X_t and Y_t contain the same information, the second representation provides greater control in the asymptotic analysis. It is easy to see that the transitions of the chain Y_t are defined by the following: let $(b, s) \in \mathbb{Z} \times \mathbb{N}$ be a state of the Markov chain Y_t, then

$$\begin{array}{ll} (b,s) \to (b, s+\Delta) & \text{with rate } \alpha_+(\Delta), \\ (b,s) \to (b, s-\Delta) & \text{with rate } \alpha_-(\Delta), \text{ where } 0 < \Delta < s, \\ (b,s) \to (b-\Delta, s+\Delta) & \text{with rate } \beta_-(\Delta), \\ (b,s) \to (b+\Delta, s-\Delta) & \text{with rate } \beta_+(\Delta), \text{ where } 0 < \Delta < s. \end{array} \tag{3}$$

Moreover, since the transition rates of Y_t depend only on the second coordinate, the spread, we can see that S_t alone is the continuous-time Markov process and has the following transition rates: suppose that at some moment the spread is $k \in \mathbb{N}$, then

$$\begin{array}{l} k \to k+\Delta \text{ with rate } \gamma_+(\Delta) = \alpha_+(\Delta) + \beta_-(\Delta), \\ k \to k-\Delta \text{ with rate } \gamma_-(\Delta) = \alpha_-(\Delta) + \beta_+(\Delta), \text{ where } 0 < \Delta < s. \end{array} \tag{4}$$

The definition of any model relies on the functional relationship between the transition rate and the increment Δ, which is typically influenced by the intensity of liquidity fluctuations. In this study, we leverage the general model (2) and its alternative representations (3), (4) to explicitly specify the transition rates governing the spread within a low-liquidity market, accounting for potential mean gaps G in the order book.

In [17], the authors provide empirical evidence showing that the cumulative probability for non-zero price returns conditioned on order size for several different ranges of market order size is surprisingly independent of the volume. The distributions for each range of volumes are roughly similar. Each curve approximately approaches a power law for large returns independent of the volume, illustrating that the key property determining large price returns is fluctuations in market impact and that the role of the volume of the order initiating a price change is minor. Large price changes caused by large orders are very rare and play an insignificant role in determining the statistical properties of price changes.

Inspired by empirical facts outlined in [17] and [33], and as briefly mentioned earlier, we propose a dynamic model for the spread process $S_t : t \geq 0$ in the presence of liquidity fluctuations, which are manifested by gaps of mean size G in filled price levels within the OB, where the preference of economic agents have a (truncated) geometric distribution. Assuming that the spread is in the state $s, s > 1$, then the transition rates in the model (4) we define as

$$\gamma_+(\Delta) = \begin{cases} \gamma_+, & \text{if } \Delta = G, \\ 0, & \text{if } \Delta \neq G, \end{cases} \quad \text{and} \quad \gamma_-(\Delta) = \begin{cases} \gamma_- p q^{\Delta-1}, & \text{if } 1 < \Delta < s, \\ \gamma_- q^{s-1}, & \text{if } \Delta = 1, \end{cases} \tag{5}$$

where γ_+, γ_- are fixed positive real numbers, $p \in (0, 1)$ and $q = 1 - p$. Note that the spread model allows for jumps to the right of the mean size G triggered by the presence of gaps in the OB. Conversely, when the spread is j, it undergoes reduction through a Poisson process with intensity λ_- distributed on ticks between the bid and the ask. This reduction follows a truncated geometric distribution c_{ij}, where $0 < j < i$. Empirically, economic agents tend to place limit orders closer to the bid and the ask, making the truncated geometric distribution a suitable approximation to describe how the spread is reduced (see Sect. 2). The use of the geometric distribution for the spread has been previously reported in the literature, as seen in other models, such as [19].
Observe that

$$\sum_{\Delta=1}^{s} \gamma_-(\Delta) = \gamma_-.$$

If $s = 1$ we define the reflection condition γ_+. In terms of the generator of continuous-time Markov chain, the process $\{S_t : t \geq 0\}$ is characterized by the Q-matrix, denoted as $Q = (q_{ij})$, which can be expressed as follows:

$$\begin{aligned} q_{ij} &= \gamma_- c_{ij} \mathbf{1}_{[1,i\rangle}(j) + \gamma_+ \mathbf{1}_{\{i+G\}}(j), \quad i \neq j, \\ q_{ii} &= -(\gamma_- + \gamma_+), \quad i \geq 1, \\ q_{1,1+G} &= \gamma_- + \gamma_+, \end{aligned} \tag{6}$$

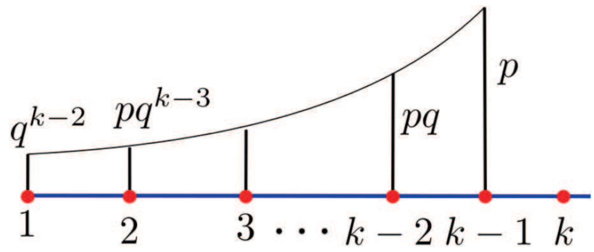

FIGURE 2. $\{c_{ki} : i = 1, 2, \ldots, k-1\}$ is a probability distribution for each $k \in \{1, 2, \ldots\}$. Orders arrive one at a time, and large price changes are caused by gaps of mean size G in the order book, where c_{ki} represent the preference of economic agents and $pq^{k-i-1}, i = 2, \ldots, k-1$ and q^{k-2}, $i = 1$.

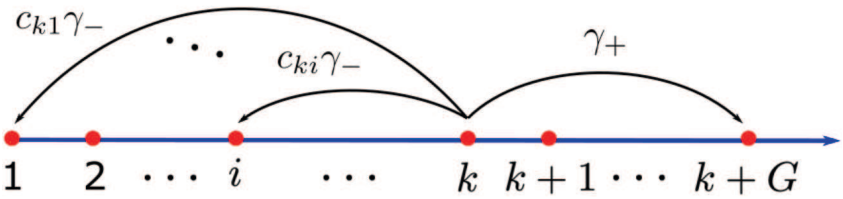

FIGURE 3. Spread model with rates $c_{ij}\gamma_-$, where $1 < j < i-1$ and $\gamma_+ > 0$.

and 0 otherwise, where for each $i \in \{2, 3, \ldots\}$ the sequence $\{c_{ij} : j = 1, 2, \ldots, i-1\}$ is a truncated geometric distribution $\left(\sum_j c_{ij} = 1\right)$ with $p + q = 1$:

$$\begin{aligned}
&\text{If } \ i = 2, \text{ then } \ c_{21} = 1 \\
&\text{If } \ i = 3, \text{ then } \ c_{31} = q, \ \ c_{32} = p \\
&\text{If } \ i = 4, \text{ then } \ c_{41} = q^2, \ \ c_{42} = pq, \ \ c_{43} = p \\
&\text{If } \ i = 5, \text{ then } \ c_{51} = q^3, \ \ c_{52} = pq^2, \ \ c_{53} = pq, \ \ c_{54} = p \\
&\quad \ldots
\end{aligned}$$

or, in short

$$c_{ij} = \begin{cases} pq^{i-j-1}, 1 < j < i \\ q^{i-2}, j = 1 \end{cases} \tag{7}$$

The jump from the state k to the state $k+G$ represents a liquidity fluctuation caused by gaps of mean size G on the left (buy limit orders) or right side (sell limit orders) in the order book.

Denote $\gamma = \gamma_- + \gamma_+$, then the Q-matrix of the spread process is following

$$Q = \begin{pmatrix}
-\gamma & 0 & 0 & 0 & 0 & \cdots \gamma & 0 & 0 & 0 & \cdots \\
c_{21}\gamma_- & -\gamma & 0 & 0 & 0 & \cdots 0 & \gamma_+ & 0 & 0 & \cdots \\
c_{31}\gamma_- & c_{32}\gamma_- & -\gamma & 0 & 0 & \cdots 0 & 0 & \gamma_+ & 0 & \cdots \\
c_{41}\gamma_- & c_{42}\gamma_- & c_{43}\gamma_- & -\gamma & 0 & \cdots 0 & 0 & 0 & \gamma_+ & \cdots \\
c_{51}\gamma_- & c_{52}\gamma_- & c_{53}\gamma_- & c_{54}\gamma_- & -\gamma & \cdots 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \vdots & \vdots & \vdots & \vdots & \cdots \\
c_{i1}\gamma_- & c_{i2}\gamma_- & c_{i3}\gamma_- & c_{i4}\gamma_- & c_{i5}\gamma_- & \cdots 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

Note that the process defined by Q is irreducible and non-explosive, guaranteeing that the chain can exhibit only a finite number of transitions within any given finite time interval.

Remark 2.1. For simplicity, this work considers γ_+ and γ_- as constants. Although this imposes an artificial constraint on spread closing events, it allows us to derive theoretical results concerning the existence of an invariant measure and facilitates considerations regarding large deviations (see Section 4 and 6). A straightforward generalization could involve allowing γ_- to depend on s in a linear form, such as $\gamma_- = \gamma_0 + \gamma_1 s$. However, describing the model in this scenario becomes challenging, particularly in a simple linear case. While such a model may potentially avoid catastrophic events, it proves to be non-solvable. Regarding large deviations, in the absence of catastrophes, authors in [38] have satisfactorily addressed this matter. See [26] for the Large Deviation Principle (LDP) in phase space. However, the LDP for the process with catastrophes remains an open question, and we intend to address it in future investigations.

3. Stability, recurrence, and transience

In this section, we discuss the stability conditions for the Markov chain S_t. Before starting, note that it is well known that studying the process (6) is equivalent to studying its embedding chain. To do this, we will begin by considering the embedding chain building from Q.

Let $\eta_n, n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, be the embedded Markov chain with state space $\mathbb{N}$ whose transition probabilities are defined from the matrix Q in the following way:

$$p(i,j) := \mathbb{P}(\eta_{n+1} = j \mid \eta_n = i) = \begin{cases} \dfrac{\gamma_+}{\gamma}, & \text{if } j = i + G, i > 1; \\ 1, & \text{if } j = 1 + G, i = 1; \\ \dfrac{\gamma_-}{\gamma} p q^{i-j-1}, & \text{if } 1 < j < i; \\ \dfrac{\gamma_-}{\gamma} q^{i-2}, & \text{if } j = 1; \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Without loss of generality, let us suppose the initial state as $\eta_0 = 1$. Let $\nu(t)$, $t \in \mathbb{R}_+$, be a Poisson process with rate γ (recall, that $\gamma = \gamma_- + \gamma_+$). We assume the Poisson process $\nu(\cdot)$ and the Markov chain η_n are independent. Thus, the continuous-time Markov chain S_t can be expressed as follows:

$$S_t = \eta(\nu(t)), \ t \in \mathbb{R}_+.$$

Note that the embedded chain has negative increments with a geometric distribution. Therefore, we expect that S_t behaves similarly to a random walk on $\mathbb{N}$ with a bias. The following criterion holds.

Theorem 3.1. *Let η_n, $n \in \mathbb{N}_0$, be the Markov chain with stochastic matrix $\mathbb{P} = (p(i,j), i, j \in \mathbb{N})$, defined in (8). Then,*

(1) η_n is positive recurrent, if $\gamma_+ G - \dfrac{\gamma_-}{p} < 0$;

(2) η_n is null recurrent, if $\gamma_+ G - \dfrac{\gamma_-}{p} = 0$;

(3) η_n is transient, if $\gamma_+ G - \dfrac{\gamma_-}{p} > 0$.

Proof. The criterion for both the continuous-time chain S_t and its embedded discrete-time chain η_n is the same. Consequently, it suffices to establish the criteria for the discrete-time chain η_n. To accomplish this, we employ the Lyapunov function technique, as described in [18] and [27]. The objective is to identify a function f defined on the state space, such that the process $f(\eta_n)$ behaves as a sub- or super-martingale.

In all cases, we will utilize the Lyapunov function $f(x) = x$. The key factor here is the expectation of the chain's increment, also known as the *drift*. For the sake of simplicity, let's assume $\gamma_+ + \gamma_- = 1$ for the notation.

$$\begin{aligned} E_k &= \mathbb{E}(\eta_1 - \eta_0 \mid \eta_0 = k) \\ &= \gamma_+ G - \gamma_- p\big(1 + 2q + \ldots + (k-2)q^{k-3}\big) - \gamma_-(k-1)q^{k-2} \\ &= \gamma_+ G - \gamma_- \left(\frac{1 - q^{k-1}}{1-q} + (2k-1)q^{k-2}\right). \end{aligned}$$

Positive recurrence (ergodicity). The ergodicity we prove by Foster criteria (see, for example, Thm. 2.6.4, pp. 63 in [27]). Indeed, using the identity function $f(x) = x$, if $\gamma_+ G - \frac{\gamma_-}{p} < 0$, then there exist a small $\varepsilon > 0$ and integer $k_0 \in \mathbb{N}$ such that, for all $k > k_0$ the condition (C.7) holds

$$E_k \le -\varepsilon.$$

In this case the set A we define as $A = \{k \in \mathbb{N} : k \le k_0\}$ and the condition (C.8) holds.

Transience. Indeed, again we use the identity (Lyapunov) function $f(x) = x$. If $\gamma_+ G - \frac{\gamma_-}{p} > 0$, then there exist a small $\varepsilon > 0$ and integer $k_0 \in \mathbb{N}$ such that, for all $k > k_0$ the condition (C.9) holds

$$E_k \ge \varepsilon.$$

In this case the set A we define as $A = \{k \in \mathbb{N} : k \le k_0\}$ and the condition (C.10) holds true:

$$\mathbb{E}\left(|f(X_{n+1}) - f(X_n)|^{1+\delta} \mid X_n = x\right) < \gamma_+ G^{1+\delta} + \gamma_- p \sum_{k=1}^{\infty} q^k k^{1+\delta} =: B < \infty,$$

for all $x \in \Sigma \setminus A$.

Null-recurrence. When $\gamma_+ G - \frac{\gamma_-}{p} = 0$ the chain η_n has asymptotically zero drift (Lamperti type): $E_k \to 0$ as $k \to \infty$. Moreover, the drift tends to zero exponentially fast. It means that essentially the behavior of η_n is similar to zero drift random walk with bounded increments, which is null-recurrent. Thus it is naturally expect that in this case η_n is null recurrent. Indeed, conferring the limits of condition **M2** (see Appendix C):

$$\begin{aligned} a &:= \lim_{k\to\infty} kE_k = 0 \\ b &:= \lim_{k\to\infty} \mathbb{E}\left((\eta_{n+1} - \eta_n)^2 \mid \eta_n = k\right) = \gamma_+ G^2 + \gamma_- p \sum_{k=1}^{\infty} q^k k^2 < \infty, \end{aligned}$$

and conferring condition **M0** and **M1** which also holds, the η_n is null recurrent if $|2a| < b$ (see Appendix C). This proves the Theorem. □

Remark 3.2. If $pG\gamma_+ = \gamma_-$ then it is easy to prove that $\frac{\eta(\nu(nt))}{\sqrt{n}}$ converges weakly as $n \to \infty$ to $\sigma|W(t)|$ in the space of càdlàg functions on the interval $[0,1]$ with Sorokhod metric, where $W(t)$ is a Wiener process. Note that $\sigma = \sigma(G, \gamma_+, \gamma_-, p)$ can be found explicitly.

4. Stationary distribution

Considering that positive recurrence in the discrete-time scenario is associated with the presence of an invariant distribution (refer to Thm. 3.5.3 on page 118 in [31]), we will adopt the parameter condition stated in Theorem 3.1 for this section: $\gamma_+ G - \dfrac{\gamma_-}{p} < 0$.

Let us denote by $p(t)$ the distribution of the spread process S_t at time t, represented as the row vector $p(t) = (p_j(t))_{j\in\mathbb{N}}$, where $p_j(t) := \mathbb{P}(S_t = j)$ for $j \in \mathbb{N}$. It is important to note that the distribution $p(t)$ is uniquely determined by the initial distribution $p(0)$ and the corresponding forward Kolmogorov equation

$$p'(t) = p(t)Q. \tag{9}$$

The elements of Q ensure that an infinite number of jumps does not occur within a finite time interval (refer to, for example, [31]). We solve the equation (9) using the generating function approach. Let

$$\phi(s,t) = \sum_{j=1}^{\infty} \mathbb{P}(S_t = j)s^j. \tag{10}$$

The next proposition establishes a differential equation for generating function ϕ derived from the Kolmogorov equation (9).

Proposition 4.1. *The generating function ϕ takes the form*

$$\frac{\partial \phi}{\partial t} = -\gamma_- s(1-s^G)p_1(t) + \left[\gamma_+ s^G - \gamma - \frac{p\gamma_-}{q-s}\right]\phi(s,t) + \gamma_- \frac{s-s^2}{q(q-s)}\phi(q,t). \tag{11}$$

Let $\Psi(s,\theta), \hat{p}_1(\theta)$ be Laplace transforms with respect to t of the functions $\phi(s,t)$ and $p_1(t)$ respectively. According to the (11) we obtain the following relation:

$$\begin{aligned}&\left(\theta - \gamma_+ s^G + \gamma + \frac{p\gamma_-}{q-s}\right)\Psi(s,\theta)\\ &\qquad = -\gamma_-\left(s - s^{G+1}\right)\hat{p}_1(\theta) + \gamma_-\left(\frac{s-p}{q} + \frac{p}{q-s}\right)\Psi(q,\theta) + \phi(s,0).\end{aligned} \tag{12}$$

We assumed that $\phi(s,0)$ is known, meaning that the initial distribution of the spread process is known.

Let $\lambda = \{\lambda_k\}_{k=1}^{\infty}$, be the stationary measure, and let $\Psi(s)$ be their generating function. The probability generating function can be found as the limit $\Psi(s) = \lim_{\theta\to 0}\theta\Psi(s,\theta)$, and due to (12) it satisfies the following relation:

$$\left(\gamma + \frac{p\gamma_-}{q-s} - \gamma_+ s^G\right)\Psi(s) = -\gamma_-\left(s - s^{G+1}\right)\lambda_1 + \gamma_-\left(\frac{s-p}{q} + \frac{p}{q-s}\right)\Psi(q). \tag{13}$$

By differentiating expression (13) and evaluating it at $s = 0$ and $s = 1$, we obtain the expressions for λ_1 and $\Psi(q)$:

$$\begin{aligned}\lambda_1 &= \frac{p}{q\gamma + (1+pG)\gamma_-}\left(\frac{\gamma_-}{p} - G\gamma_+\right),\\ \Psi(q) &= \frac{q^2}{\gamma_-}\left(\gamma + \frac{\gamma_-}{q}\right)\frac{p}{q\gamma + (1+pG)\gamma_-}\left(\frac{\gamma_-}{p} - G\gamma_+\right).\end{aligned} \tag{14}$$

It is well known that the probabilities of the invariant distribution can be derived by evaluating the derivatives at zero of their generating function Ψ:

$$\lambda_k = \frac{1}{k!}\frac{d^k\Psi}{ds^k}(0).$$

Indeed, let us represent the equation (13) in the form

$$A(s)\Psi(s) = \gamma_-\Psi(q)B(s) - \gamma_-\lambda_1 C(s), \tag{15}$$

with

$$A(s) = \gamma + \frac{p\gamma_-}{q-s} - \gamma_+ s^G$$
$$B(s) = \frac{s-p}{q} + \frac{p}{q-s}$$
$$C(s) = s - s^{G+1}.$$

The n-th derivative of (13) at $s=0$ is given by

$$\sum_{k=0}^{n} \binom{n}{k} \Psi^{(k)}(0) A^{(n-k)}(0) = \gamma_- \Psi(q) B^{(n)}(0) - \gamma_- \lambda_1 C^{(n)}(0),$$

with k-th derivatives

$$A^{(k)}(0) = \frac{p}{q^{k+1}} k! - \gamma_+ G! \gamma_- \mathbf{1}_{\{k=G\}}$$
$$B^{(k)}(0) = \frac{p}{q^{k+1}} k! + \frac{1}{q} \mathbf{1}_{\{k=1\}}$$
$$C^{(k)}(0) = \mathbf{1}_{\{k=1\}} - (G+1)! \mathbf{1}_{\{k=G+1\}}.$$

By differentiating equation (13) and evaluating it at $s=0$, we can derive the following system of equations that encompass all invariant probabilities $\lambda_k, k=1,2,\ldots$:

$$\begin{aligned} \left(\gamma + \frac{\gamma_-}{q}\right)\lambda_1 &= \frac{\gamma_-}{q^2}\Psi(q) \\ \sum_{i=1}^{k-1} \frac{p\gamma_-}{q^{k-i+1}}\lambda_i + \left(\gamma_+ + \frac{\gamma_-}{q}\right)\lambda_k &= \frac{p\gamma_-}{q^{k+1}}\Psi(q), \quad 2 \le k \le G \\ \sum_{i=1}^{G} \left(\frac{p\gamma_-}{q^{G+2-i}} - \gamma \mathbf{1}_{\{i=1\}}\right)\lambda_i + \left(\gamma_+ + \frac{\gamma_-}{q}\right)\lambda_{G+1} &= \frac{p\gamma_-}{q^{G+2}}\Psi(q), \; k = G+1 \\ \sum_{i=1}^{k-1} \left(\frac{p\gamma_-}{q^{k-i+1}} - \gamma_+ \mathbf{1}_{\{i=k-G\}}\right)\lambda_i + \left(\gamma_+ + \frac{\gamma_-}{q}\right)\lambda_k &= \frac{p\gamma_-}{q^{k+1}}\Psi(q), \quad k > G+1 \end{aligned} \tag{16}$$

where the second line is omitted when $G=1$. If we denote by

$$d = \gamma + \frac{\gamma_-}{q}, \quad D = \frac{\gamma_+}{p\gamma_-} + \frac{1}{pq}, \quad N = \frac{1}{q^{G+1}} - \frac{\gamma}{p\gamma_-} \quad , \quad M = \frac{1}{q^{G+1}} - \frac{\gamma_+}{p\gamma_-}$$

the system of equations (16) can be expressed as follows:

$$\begin{aligned} \lambda_1 &= \frac{\gamma_-}{q^2 d}\Psi(q) \\ \frac{\lambda_1}{q^k D} + \frac{\lambda_2}{q^{k-1} D} + \cdots + \frac{\lambda_{k-1}}{q^2 D} + \lambda_k &= \frac{1}{q^{k+1}} \frac{\Psi(q)}{D}, \quad 2 \le k \le G \\ \frac{N}{D}\lambda_1 + \frac{\lambda_2}{q^G D} + \frac{\lambda_3}{q^{G-1} D} + \cdots + \frac{\lambda_G}{q^2 D} + \lambda_{G+1} &= \frac{1}{q^{G+2}} \frac{\Psi(q)}{D}, \quad k = G+1 \\ \frac{\lambda_1}{q^k D} + \frac{\lambda_2}{q^{k-1} D} + \cdots + \frac{M}{D}\lambda_{k-G} + \cdots + \frac{\lambda_{k-1}}{q^2 D} + \lambda_k &= \frac{1}{q^{k+1}} \frac{\Psi(q)}{D}, \quad k > G+1 \end{aligned} \tag{17}$$

When we set G to 1, this system can be expressed in a simpler form as follows:

$$\begin{aligned} \lambda_1 &= \frac{\gamma_-}{q^2 d}\Psi(q) \\ -\frac{\gamma\lambda_1}{p\gamma_- D} + \frac{\lambda_1}{q^2 D} + \lambda_2 &= \frac{1}{q^3}\frac{\Psi(q)}{D}, \\ -\frac{\gamma_+\lambda_2}{p\gamma_- D} + \frac{\lambda_1}{q^3 D} + \frac{\lambda_2}{q^2 D} + \lambda_3 &= \frac{1}{q^4}\frac{\Psi(q)}{D}, \\ -\frac{\gamma_+\lambda_3}{p\gamma_- D} + \frac{\lambda_1}{q^4 D} + \frac{\lambda_2}{q^3 D} + \frac{\lambda_3}{q^2 D} + \lambda_4 &= \frac{1}{q^5}\frac{\Psi(q)}{D}, \\ &\cdots \end{aligned} \tag{18}$$

Note that the system (17) can be written in the form

$$\left(\mathbf{I} + \frac{1}{D}\mathbf{N}\right)\lambda = \Psi(q)b, \tag{19}$$

where

$$\mathbf{N} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ x^2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ x^3 & x^2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ x^4 & x^3 & x^2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ x^5 & x^4 & x^3 & x^2 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots \\ x^G & x^{G-1} & x^{G-2} & x^{G-3} & x^{G-4} & \cdots & x^2 & 0 & 0 & 0 & \cdots \\ N & x^G & x^{G-1} & x^{G-2} & x^{G-3} & \cdots & x^3 & x^2 & 0 & 0 & \cdots \\ x^{G+2} & M & x^G & x^{G-1} & x^{G-2} & \cdots & x^4 & x^3 & x^2 & 0 & \cdots \\ x^{G+3} & x^{G+2} & M & x^G & x^{G-1} & \cdots & x^5 & x^4 & x^3 & x^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad b = \begin{pmatrix} \dfrac{x^2\gamma_-}{d} \\ \dfrac{x^3}{D} \\ \dfrac{x^4}{D} \\ \vdots \\ \dfrac{x^{G+1}}{D} \\ \dfrac{x^{G+2}}{D} \\ \vdots \end{pmatrix}$$

with $x = 1/q$, and $\Psi(q)$ is given by (14). Thanks of the matrix representation (19) we immediately have the following theorem.

Theorem 4.2. *If $\gamma_+ G - \dfrac{\gamma_-}{p} < 0$, then the stationary probability distribution $\lambda = \{\lambda_k\}$ can be written in the form*

$$\lambda = \left(b + \sum_{k=1}^{\infty} \frac{(-1)^k}{D^k}\mathbf{N}^k b\right)\Psi(q). \tag{20}$$

Proof. Indeed, note that in representation (19) the matrix $\mathbf{N}$ is a strictly triangular matrix (with zeros on the diagonal), and it satisfies the relation $\lim_{n\to\infty}\mathbf{N}^n = 0$. Therefore, we can use the polynomial identity $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$, and using $x = -\mathbf{N}$, we obtain the matrix relation (20). This proves the Theorem. ☐

It is evident that the solution (20) can be explicitly written for any given G, but the formulas become complex for arbitrary G. In order to calculate the drift in the Law of Large Numbers, we require knowledge of the measure to compute the mean value within this context. Fortunately, we won't need the entire exact formula for λ; we will only require λ_1 and $\Psi(q)$, which are provided by (14).

5. Law of Large Numbers (LLN) and Central Limit Theorem (CLT).

Denote by (b_n, s_n) the embedding Markov chain of the bid-spread process (B_t, S_t), described by the transition rates (3). Let $\mu = \{\mu_k\}_{k=1}^{\infty}$ be the stationary distribution for the chain s_n. Note that the invariant measure λ for the continuous-time spread $\{S_t\}$, computed in Theorem 4.2, is related to μ by simple relation $\mu_k = \lambda_k \gamma$ (remembering that $\gamma = \gamma_- + \gamma_+$).

Given a change in the spread and utilizing the transition rates (3), the following equations quantify the probability that the observed spread change can be attributed to a change in either the bid or ask:

$$\begin{aligned}\mathbb{P}\left(\Delta b_n > 0 \mid \Delta s_n < 0\right) &= 1 - \mathbb{P}\left(\Delta b_n = 0 \mid \Delta s_n < 0\right) = \frac{\beta_+}{\gamma_-},\\ \mathbb{P}\left(\Delta b_n = -G \mid \Delta s_n = G\right) &= 1 - \mathbb{P}\left(\Delta b_n = 0 \mid \Delta s_n = G\right) = \frac{\beta_-}{\gamma_+},\end{aligned} \tag{21}$$

where $\Delta b_n := b_n - b_{n-1}$ and $\Delta s_n := s_n - s_{n-1}$.

Using the relations (21) the dynamics of the embedding chain $\{b_n\}$ can be expressed in terms of the spread dynamics $\{s_n\}$ as follows:

$$b_n = \sum_{k=1}^{n} F(s_{k-1}, s_k, U_k) \tag{22}$$

where $\{U_n\}$ is a sequence of iid uniform random variables, and function F is defined as follows:

$$F(x,y,u) = \begin{cases} -G, \text{ if } y = x + G \text{ and } u \le \dfrac{\beta_-}{\gamma_+}, \\ x - y, \text{ if } y < x \text{ and } u \le \dfrac{\beta_+}{\gamma_-}, \\ 0, \quad \text{otherwise.} \end{cases}$$

The Ergodic Theorem for Markov chains ensures the Law of Large Numbers (LLN) for the chain $\{b_n\}$.

Theorem 5.1. *Under the condition $\gamma_+ G - \dfrac{\gamma_-}{p} < 0$, the strong LLN for the chain $\{b_n\}$ holds true*

$$\frac{b_n}{n} \to \mathbb{E}_{\hat{\mu}}(F) \quad a.s. \tag{23}$$

as $n \to \infty$, where $\hat{\mu}$ is the invariant measure of the chain $\{\hat{r}_n := (s_{n-1}, s_n, U_n)\}$, and the drift $\mathbb{E}_{\hat{\mu}}(F)$ can be calculated explicitly

$$\mathbb{E}_{\hat{\mu}}(F) = \frac{1}{\gamma}\left(\frac{\beta_+}{p} - G\beta_-\right) - \lambda_1\left(\frac{G\beta_-\gamma_-}{\gamma_+} + \frac{\beta_+}{p}\Big(3 + \frac{q\gamma}{p\gamma_-}\Big)\right),$$

where λ_1 is given by (14).

The formula for the drift we deduced in the appendix.

The next step of our analysis is the establishment of the Central Limit Theorem (CLT) as it pertains to the bid price. The utilization of Harris's ergodic theorem is applicable in this context. The sole requirement for its application is the finite variance of F under the invariant measure $\hat{\mu}$.

Theorem 5.2. *If $\{p_n\}$ the jump chains of the bid price $\{B_t\}$ under the condition $\gamma_+ G - \frac{\gamma_-}{p} < 0$. Then, $\sigma_H^2 = Var_{\hat{\mu}}(F) < \infty$, and*

$$\sqrt{n}\,(\hat{p}_n - \mathbb{E}_{\hat{\mu}}(F)) \to N(0, \sigma_H^2) \tag{24}$$

as $n \to \infty$, where $\hat{\mu}$ is the invariant measure of the chain $\hat{r}_n = (s_{n-1}, s_n, U_n)$ and $\mathbb{E}_{\hat{\mu}}(F)$ is given by Theorem 5.1.

Proof. It is easy to show that $\sigma_H^2 = Var_{\hat{\mu}}(F) < \infty$ using (B.5) and (B.6). To establish the proof of equation (24), we employ the CLT to Harris ergodic Markov chain as outlined in Theorem 9 of [24]. □

Finally, the combination of Theorems 5.1 and 5.2 establishes that prices exhibit a diffusive behavior centered around a local drift $\mathbb{E}_{\hat{\mu}}(F)$, with a finite diffusion coefficient represented by $\sqrt{\sigma_H^2}$. Unfortunately, deriving an explicit form for σ_H in our specific case is not possible; however, it can be obtained through numerical simulations.

6. Local large deviations principle for positive excursions

In this section, we will be interested in the local large deviation principle (LLDP) for positive excursions of the family of processes

$$S_T(t) := \frac{S_{Tt}}{T},\ t \in [0,1], \tag{25}$$

here $T > 0$ is an increasing scaling parameter. We will consider the trajectories of the process $S_T(\cdot)$ on the space of càdlàg functions $\mathbb{D}[0,1]$. Define a uniform metric on this space

$$\rho(f,g) := \sup_{t\in[0,1]} |f(t) - g(t)|,\ f,g \in \mathbb{D}[0,1].$$

Let $\mathfrak{B}$ denote the Borel σ - algebra in metric space $(\mathbb{D}[0,1],\rho)$. Recall the definition of the LLDP.

Definition 1. *Let $\mathbb{G} \subseteq \mathbb{D}[0,1]$ and $\mathbb{G} \in \mathfrak{B}$. We say that family of the processes $S_T(\cdot)$ satisfies an $\mathbb{G}$-LLDP with a rate function $I : \mathbb{G} \to [0,\infty]$ and the normalizing function $\psi(T)$ if for all $f \in \mathbb{G}$*

$$\begin{aligned}\lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}(S_T(\,\cdot\,) \in \mathbb{U}_\varepsilon(f))\\ = \lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{\psi(T)} \ln \mathbb{P}(S_T(\,\cdot\,) \in \mathbb{U}_\varepsilon(f)) = -I(f),\end{aligned}$$

where

$$\mathbb{U}_\varepsilon(f) = \{g \in \mathbb{D}[0,1] :\ \rho(f,g) < \varepsilon\}.$$

Let $\mathbb{V}[0,1]$ be the set of continuous functions with a finite variation on the interval $[0,1]$. Every function $f \in \mathbb{V}[0,1]$ has a unique decomposition into absolutely continuous and singular components

$$f(t) = f_a(t) + f_s(t),\ f_a(0) = f(0),\ f_s(0) = 0.$$

Further, the function $f_s(\cdot)$ has a unique decomposition into monotone increasing and decreasing components

$$f_s(t) = f_s^+(t) - f_s^-(t),\ f_s^+(0) = f_s^-(0) = 0.$$

So, we have

$$f(t) = f_a(t) + f_s^+(t) - f_s^-(t).$$

For more details about these decompositions, refer to see Chapter 1, Section 4 of [34] and Chapter 9, Section 6 of [30]. Now, we proceed to define the following sets of functions

$$\begin{aligned}\mathbb{A} &:= \{f \in \mathbb{V}[0,1] : f(0) = 0,\ f_s^+(t) \equiv 0\},\\ \mathbb{G} &:= \{f \in \mathbb{V}[0,1] : f_a \in \mathbb{L}[0,1], f(t) > 0 \text{ for any } t \in (0,1]\},\end{aligned}$$

here $\mathbb{L}[0,1]$ is the set of Lipschitz functions on the interval $[0,1]$.

Let us introduce additional notations

$$A(\lambda) := \gamma_+ e^{\lambda G} + \gamma_- p \frac{e^{-\lambda}}{1 - qe^{-\lambda}} - \gamma,$$
$$\Lambda(y) := \sup_{\lambda\in\mathbb{R}} (\lambda y - A(\lambda)),\ y \in \mathbb{R}.$$

$\Lambda(y)$ is a Legendre transform of a function $A(\lambda)$.

Subsequently, the following representation of S_t will prove to be useful. Consider two independent Poisson processes ν_1 and ν_2 with rates γ_+ and γ_-, respectively. The spread process S_t, defined by the Q matrix (6), can be expressed in terms of ν_1 and ν_2 as follows: let (t_k) denote the jump instances of ν_2, then

$$S_t = G\nu_1(t) - \sum_{k=1}^{\nu_2(t)} \xi_k(S_{t_k^-}). \tag{26}$$

In this representation, we introduce a family of integer-valued independent random variables (downward jumps) $\xi_i(m)$, where $i \in \mathbb{Z}+$ and $m \in \mathbb{N}$. This family is independent of ν_1 and ν_2. For a given m, the random variables $\xi_i(m)$ follow a truncated geometric distribution as defined in (7):

$$\mathbb{P}(\xi_i(m) = k) = c_{m,m-k}.$$

The following theorem is the main result of this section.

Theorem 6.1. *The family of the processes $S_T(\cdot)$ satisfies a $\mathbb{G}$-LLDP with the normalizing function $\psi(T) = T$ and the rate function*

$$I(f) = \begin{cases} \displaystyle\int_0^1 \Lambda(f_a'(t))dt - (\ln q) f_s^-(1),\ f \in \mathbb{G} \cap \mathbb{A}, \\ \infty,\ f \in \mathbb{G} \setminus \mathbb{A}. \end{cases}$$

Proof. One of the main component in the proof is Lemma D.1 that establishes the $\mathbb{G}$-LLDP for the auxiliary process $\hat{S}$ defined as the following compound Poisson process

$$\hat{S}_t := \sum_{k=1}^{\nu(t)} \hat{\zeta}_k,\ t \in \mathbb{R}_+,$$

where i.i.d. random variables $\hat{\zeta}_1, \ldots, \hat{\zeta}_k, \ldots$ don't depend on Poisson process $\nu(\cdot)$ and

$$\mathbb{P}(\hat{\zeta}_1 = G) = \frac{\gamma_+}{\gamma},\ \mathbb{P}(\hat{\zeta}_1 = -l) = \frac{\gamma_-}{\gamma} pq^{l-1},\ l \in \mathbb{N}.$$

Note that the process $\hat{S}$ experiences unbounded downward jumps; however, these substantial jumps occur with geometrically small probabilities. As a result, when applying the large deviations technique, the asymptotic behavior of both processes aligns. Furthermore, Lemma D.1 (see Appendix) confirms that both processes share the same rate function.

Let us first show an upper-bound

$$\lim_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{1}{T} \ln \mathbb{P}(S_T(\,\cdot\,) \in \mathbb{U}_\varepsilon(f)) \le -I(f), \tag{27}$$

for any $f \in \mathbb{G}$. Denote

$$r_{1,\varepsilon,T} := \lfloor (f(\delta) - \varepsilon)T \rfloor,\ r_{2,\varepsilon,T} := \lfloor (f(\delta) + \varepsilon)T \rfloor.$$

Since $f \in \mathbb{G}$, using Lemma D.2, we have for any $\delta \in (0,1)$, there exist sufficiently small $\varepsilon > 0$ and sufficiently large T such that

$$\begin{aligned}
\mathbb{P}(S_T(\,\cdot\,) \in \mathbb{U}_\varepsilon(f)) &\le \mathbb{P}\left(\sup_{t\in[\delta,1]} |S_T(t) - f(t)| < \varepsilon\right) \\
&= \sum_{r=r_{1,\varepsilon,T}}^{r_{2,\varepsilon,T}} \mathbb{P}\left(\sup_{t\in[\delta,1]} |S_T(t) - f(t)| < \varepsilon \;\middle|\; S(T\delta) = r\right) \mathbb{P}(S_{T\delta} = r) \\
&\le (2\varepsilon T + 1) \max_{r_{1,\varepsilon,T}\le r\le r_{2,\varepsilon,T}} \mathbb{P}\left(\sup_{t\in[\delta,1]} |S_T(t) - f(t)| < \varepsilon \;\middle|\; S_{T\delta} = r\right) \\
&= (2\varepsilon T + 1) \max_{r_{1,\varepsilon,T}\le r\le r_{2,\varepsilon,T}} \mathbb{P}\left(\sup_{t\in[\delta,1]} |\hat{S}_T(t) - f(t)| < \varepsilon \;\middle|\; \hat{S}_{T\delta} = r\right) \\
&\le (2\varepsilon T + 1) \max_{r_{1,\varepsilon,T}\le r\le r_{2,\varepsilon,T}} \mathbb{P}\left(\sup_{t\in[\delta,1]} |(\hat{S}_T(t) - \hat{S}_T(\delta)) - (f(t) - f(\delta))| < \varepsilon \;\middle|\; \hat{S}_{T\delta} = r\right) \\
&= (2\varepsilon T + 1)\mathbb{P}\left(\sup_{t\in[\delta,1]} |(\hat{S}_T(t) - \hat{S}_T(\delta)) - (f(t) - f(\delta))| < \varepsilon\right) \\
&= (2\varepsilon T + 1)\mathbb{P}\left(\sup_{u\in[0,1]} |(\hat{S}_T(u(1-\delta)+\delta) - \hat{S}_T(\delta)) - (f(u(1-\delta)+\delta) - f(\delta))| < \varepsilon\right) \\
&= (2\varepsilon T + 1)\mathbb{P}\left(\sup_{u\in[0,1]} |(\hat{S}_T((1-\delta)u) - \tilde{f}(u)| < \varepsilon\right),
\end{aligned}$$

where $\hat{S}_T$, defined by (D.11), is the same scaling as (25) applied for $\hat{S}_t$, and

$$u := \frac{t-\delta}{1-\delta}, \;\; \tilde{f}(u) := f((1-\delta)u + \delta) - f(\delta). \tag{28}$$

Random process $\hat{S}((1-\delta)t)$ as a process on time t can be considered as a compound Poisson process with rate $(1-\delta)\gamma$. Therefore, it follows from the Lemmas D.1, D.3 that for any $\delta > 0$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{T\to\infty} \frac{1}{T} \ln\left((2\varepsilon T + 1)\mathbb{P}\left(\sup_{u\in[0,1]} |(\hat{S}_T((1-\delta)u) - \tilde{f}(u)| < \varepsilon\right)\right) \\
&= \begin{cases} -(1-\delta)\displaystyle\int_0^1 \Lambda\left(\frac{\tilde{f}_a'(t)}{1-\delta}\right) dt + (\ln q)\tilde{f}_s^-(1), \;\; \tilde{f} \in \mathbb{G}\cap\mathbb{A}, \\ -\infty, \;\; \tilde{f} \in \mathbb{G}\setminus\mathbb{A}. \end{cases} \\
&= \begin{cases} -\displaystyle\int_\delta^1 \Lambda(f_a'(t))dt + (\ln q)(f_s^-(1) - f_s^-(\delta)), \;\; \tilde{f} \in \mathbb{G}\cap\mathbb{A}, \\ -\infty, \;\; \tilde{f} \in \mathbb{G}\setminus\mathbb{A}. \end{cases}
\end{aligned} \tag{29}$$

Using the equality (29) and passing to the limit $\delta \to 0$, we obtain the desired upper-bound (27).

Let us now show that a lower-bound

$$\lim_{\varepsilon\to 0}\liminf_{T\to\infty} \frac{1}{T} \ln \mathbb{P}(S_T(\,\cdot\,) \in \mathbb{U}_\varepsilon(f)) \ge -I(f) \tag{30}$$

holds for any $f \in \mathbb{G}$.

If $f \in \mathbb{G} \setminus \mathbb{A}$ then $I(f) = \infty$ and inequality (30) is obviously true. Consider $f \in \mathbb{G} \cap \mathbb{A}$. For a given $\delta \in (0,1)$, we define the following sequence (u_l)

$$u_0 := 0,$$
$$u_l := \min\{u_{l-1} < t < \delta : |f(t) - f(u_{l-1})| = \varepsilon/2\} \wedge \delta, \; l \geq 1,$$

here we assume $\min \emptyset = \infty$. We also denote

$$l_\delta := \min\{l : u_l = \delta\},$$

the number of elements in the sequence $(u_l, l = 1, \ldots, l_\delta)$. By virtue of the continuity of the function $f(\cdot)$, such l_δ is finite. We define the events

$$\begin{aligned}
B_{l,1} &:= \{\nu_1(Tu_l) - \nu_1(Tu_{l-1}) = \lfloor \frac{T\,(f(u_l) - f(u_{l-1}))}{G} \rfloor\},\\
B_{l,2} &:= \{\nu_2(Tu_l) - \nu_2(Tu_{l-1}) = 0\},\\
C_{l,1} &:= \{\nu_1(Tu_l) - \nu_1(Tu_{l-1}) = 0\},\\
C_{l,2} &:= \{\nu_2(Tu_l) - \nu_2(Tu_{l-1}) = 1\} \cap \{\xi_{\nu_2(Tu_l)}(S_{t_{k_l}^-}) = \lfloor Tf(u_l) \rfloor - \lfloor Tf(u_{l-1}) \rfloor\},
\end{aligned}$$

Here $t_{k_l} \in [Tu_{l-1}, Tu_l]$ is the moment of time in which the process $\nu_2(t)$ has a jump. We put

$$A_l := \begin{cases} B_{l,1} \cap B_{l,2}, & \text{if } f(u_l) - f(u_{l-1}) = \varepsilon/2,\\ C_{l,1} \cap C_{l,2}, & \text{if } f(u_l) - f(u_{l-1}) = -\varepsilon/2. \end{cases}$$

Since $f \in \mathbb{G} \cap \mathbb{A}$, using Lemma D.2 and Markov property, we have for any $\delta \in (0,1)$, sufficiently small $\varepsilon > 0$ and sufficiently large T

$$\begin{aligned}
\mathbb{P}(S_T(\,\cdot\,) \in \mathbb{U}_\varepsilon(f)) &\geq \mathbb{P}\left(\sup_{t\in[\delta,1]} |S_T(t) - f(t)| < \varepsilon, \bigcap_{l=1}^{l_\delta} A_l\right)\\
&= \mathbb{P}\left(\sup_{t\in[\delta,1]} |S_T(t) - f(t)| < \varepsilon \;\middle|\; S_{T\delta} = \lfloor Tf(\delta) \rfloor\right) \mathbb{P}\left(\bigcap_{l=1}^{l_\delta} A_l\right)\\
&= \mathbb{P}\left(\sup_{t\in[\delta,1]} |\hat{S}_T(t) - f(t)| < \varepsilon \;\middle|\; \hat{S}_{T\delta} = \lfloor Tf(\delta) \rfloor\right) \mathbb{P}\left(\bigcap_{l=1}^{l_\delta} A_l\right)\\
&\geq \mathbb{P}\left(\sup_{u\in[0,1]} |(\hat{S}_T((1-\delta)u) - \tilde{f}(u)| < \frac{\varepsilon}{2}\right) \mathbb{P}\left(\bigcap_{l=1}^{l_\delta} A_l\right),
\end{aligned} \tag{31}$$

see formula (28) for the definition of u and $\tilde{f}(\cdot)$.

Using Lemmas D.1, D.3, D.4 and formulas (29), (31), we obtain for any $\delta \in (0,1)$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{1}{T} \ln \mathbb{P}(S_T(\,\cdot\,) \in \mathbb{U}_\varepsilon(f))\\
&\quad \geq -\int_\delta^1 \Lambda(f_a'(t))dt + (\ln q)(f_s^-(1) - f_s^-(\delta)) - 4M \max(\mathrm{Var} f_{[0,\delta]}, \delta),
\end{aligned} \tag{32}$$

here $\mathrm{Var} f_{[0,\delta]}$ is the total variation of f on the interval $[0,\delta]$.
Using the inequality (32) and passing to the limit $\delta \to 0$, we obtain the inequality (30) for $f \in \mathbb{G} \cap \mathbb{A}$. $\square$

Remark 6.2. Note that it is impossible to obtain the large deviation principle (LDP) in the entire space $(\mathbb{D}[0,1],\rho)$ even for a family of processes with independent increments $\hat{S}_T(\cdot)$. In order to obtain the so-called extended LDP, one considers an incomplete space $(\mathbb{D}[0,1],\rho_B)$, where ρ_B is Borovkov metric (see for more details [28, 29]).

Remark 6.3. Unfortunately, it is possible to find an explicit form of the function $\Lambda(\cdot)$ only for cases where $G \in \{1,2\}$. In these cases, it all comes down to solving third and fourth-degree equations, respectively.

Remark 6.4. If $pG\gamma_+ < \gamma_-$ then it is easy to prove using Theorem 6 that for any $c > 0$

$$\mathbf{P}\left(\sup_{t\in[0,T]} S(t) \geq cT\right) \geq \exp\left\{-T\Lambda(c)(1+o(1))\right\},$$

as $T \to \infty$.

7. Concluding remarks

Building on the insights from [17], this study introduces a model designed to address transient liquidity crises that lead to gaps within the Order Book (OB). It is crucial to emphasize that our model specifically focuses on a subset of liquidity crises and does not encompass various other regimes discussed in the literature [17, 19]. We present a straightforward model for spread dynamics that incorporates liquidity fluctuations and undertake an in-depth theoretical analysis of the model's properties, providing rigorous proofs for several key asymptotic theorems. We complement the discussion of Secs. 2-6 with the following two concluding remarks:

Remark 7.1. To obtain our results, we simplify the model by assuming the rate of spread closing (*i.e.*, how quickly the bid-ask gap narrows) is independent of its size. However, it's more realistic to consider dependence on the spread itself, like $\gamma_+ + \gamma_- s$. Unfortunately, this introduces mathematical challenges in proving the existence of a stable equilibrium state (invariant measure) and analyzing the behavior of extreme events (large deviations principle) for our model. While previous work [26, 38] addressed similar issues in different contexts, incorporating spread-size dependence remains an open theoretical question that we plan to explore in future studies. Understanding the impact of this dependence could yield valuable insights into the dynamics of order books during liquidity crises.

Remark 7.2. Furthermore, theoretical results, such as Section 6.1, can hold practical significance. For instance, we can estimate the model parameters using a sufficiently large sample of spreads. Subsequently, we can apply Section 6.1 to obtain a lower estimate of the probability that the maximum spread over an extended period will surpass a given level a. This probability estimation involves applying the rate function formula for the linear (on time) function, starting from zero and reaching a value of a at the end of the time interval. While the calculations can be explicit for $G = 1$ or $G = 2$, computational methods are necessary for the case where $G \geq 3$ to calculate the value of the rate functional. This particular aspect also requires further research, and we plan to address it in future investigations.

Appendix A. Proof of Proposition 4.1

Recall, that $\phi(s,t)$ is the generating function of S_t:

$$\phi(s,t) = \sum_{j=1}^{\infty} \mathbb{P}(S_t = j)s^j = \sum_{j=1}^{\infty} p_j(t)s^j.$$

Here, using the forward Kolmogorov equation (9) we find the derivative over t for the generating function. Multiplying the j-th component on each side of (9) by s^j and summing over j, we obtain

$$\sum_{j=1}^{\infty} p_j^{'}(t)s^j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} p_i(t)q_{ij}s^j = \sum_{j=1}^{\infty}\sum_{i=1,i\neq j}^{\infty} p_i(t)q_{ij}s^j + \sum_{j=1}^{\infty} p_j(t)q_{jj}s^j$$

The second sum can be written as follows

$$\sum_{j=1}^{\infty} p_j(t)q_{jj}s^j = -\gamma\phi(s,t) \tag{A.1}$$

The first sum can be written as follows

$$\begin{aligned}
&\sum_{j=1}^{\infty}\sum_{i=1,i\neq j}^{\infty} p_i(t)q_{ij}s^j \\
&\quad= \sum_{j=1}^{\infty}\sum_{i=1,i\neq j}^{\infty} p_i(t)\left[\gamma_- c_{ij}\mathbf{1}_{[1,i)}(j) + \gamma_+\mathbf{1}_{\{i+G\}}(j) + \gamma_-\mathbf{1}_{\{1+G\}}(j)\right]s^j \\
&\quad= \gamma_-\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} p_i(t)c_{ij}s^j + \gamma_+\sum_{j=G+1}^{\infty} p_{j-G}(t)s^j + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} p_i(t)c_{ij}s^j + \gamma_+ s^G\sum_{j=1}^{\infty} p_j s^j + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty} p_i(t)c_{ij}s^j + \gamma_+ s^G\phi(s,t) + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\sum_{i=2}^{\infty} p_i(t)\sum_{j=1}^{i-1} c_{ij}s^j + \gamma_+ s^G\phi(s,t) + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\sum_{i=2}^{\infty} p_i(t)\left[q^{i-2}s + \sum_{j=2}^{i-1} pq^{i-j-1}s^j\right] + \gamma_+ s^G\phi(s,t) + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\sum_{i=2}^{\infty} p_i(t)q^{i-1}\left[s - p + p\cdot\frac{q^i - s^i}{q-s}\cdot\frac{1}{q^{i-1}}\right] + \gamma_+ s^G\phi(s,t) + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\sum_{i=2}^{\infty} p_i(t)q^{i-1}\left[s - p + \frac{pq}{q-s} - \frac{ps^i}{(q-s)q^{i-1}}\right] + \gamma_+ s^G\phi(s,t) + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\left(\frac{s-p}{q} + \frac{p}{q-s}\right)\sum_{i=2}^{\infty} p_i(t)q^i - \frac{p\gamma_-}{q-s}\sum_{i=2}^{\infty} p_i(t)s^i + \gamma_+ s^G\phi(s,t) + \gamma_- p_1(t)s^{G+1} \\
&\quad= \gamma_-\left(\frac{s-p}{q} + \frac{p}{q-s}\right)[\phi(q,t) - qp_1(t)] - \frac{p\gamma_-}{q-s}[\phi(s,t) - sp_1(t)] + \gamma_+ s^G\phi(s,t) + \gamma_- p_1(t)s^{G+1}
\end{aligned}$$

Using before expression we have

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{i=1,i\neq j}^{\infty} p_i(t)q_{ij}s^j &= -\gamma_-(s - s^{G+1})p_1(t) \\
&\quad + \left[\gamma_+ s^G - \frac{p\gamma_-}{q-s}\right]\phi(s,t) + \gamma_-\left(\frac{s-p}{q} + \frac{p}{q-s}\right)\phi(q,t)
\end{aligned} \tag{A.2}$$

Summing (A.1) and (A.2) we obtain

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= -\gamma_- s(1-s^G)p_1(t) + \left[\gamma_+ s^G - \gamma - \frac{p\gamma_-}{q-s}\right]\phi(s,t) + \gamma_- \frac{s-s^2}{q(q-s)}\phi(q,t) \\ &= -\gamma_- s(1-s^G)p_1(t) - \left[\gamma_+(1-s^G) + \gamma_- \frac{1-s}{q-s}\right]\phi(s,t) + \gamma_- \frac{s-s^2}{q(q-s)}\phi(q,t)\end{aligned} \tag{A.3}$$

This proves Proposition 4.1. □

Appendix B. Proof of Theorem 5.1 and Theorem 5.2: drift and their second moment calculations

We need only to compute $\mathbb{E}_{\hat{\mu}}(F)$, where $\hat{\mu}$ is the invariant measure of the Markov chain $\hat{r}_n = (s_{n-1}, s_n, U_n)$. Then

$$\begin{aligned}\mathbb{E}_{\hat{\mu}}(F) &= -G\mu_1 \frac{\beta_-}{\gamma_+} - G\sum_{i=2}^{\infty} \mu_i p(i, i+G)\frac{\beta_-}{\gamma_+} + \sum_{i=2}^{\infty}\sum_{j=1}^{i-1}(i-j)\mu_i p(i,j)\frac{\beta_+}{\gamma_-} \\ &= -\frac{G\beta_-}{\gamma_+}\mu_1 - \frac{G\beta_-}{\gamma}\sum_{i=2}^{\infty}\mu_i + \frac{\beta_+}{\gamma_-}\sum_{i=2}^{\infty} i\mu_i \sum_{j=1}^{i-1} p(i,j) - \frac{\beta_+}{\gamma_-}\sum_{i=2}^{\infty}\mu_i \sum_{j=1}^{i-1} jp(i,j) \\ &= -\frac{G\beta_-}{\gamma}\left(1 + \mu_1\frac{\gamma_-}{\gamma_+}\right) + \frac{\beta_+}{\gamma}\sum_{i=2}^{\infty} i\mu_i - \frac{\beta_+}{\gamma_-}\sum_{i=2}^{\infty}\mu_i \sum_{j=1}^{i-1} jp(i,j)\end{aligned} \tag{B.4}$$

where $p(i,j) = \mathbb{P}(p_{n+1} = j \mid p_n = i)$ are defined by (8). It is straightforward to show that for $i \geq 2$

$$\sum_{j=1}^{i-1} jp(i,j) = \frac{\gamma_-}{\gamma}\left(q^{i-2} + \sum_{j=2}^{i-1} jpq^{i-j-1}\right) = \frac{\gamma_-}{\gamma}\left(q^{i-1} + i - \frac{1-q^i}{p}\right). \tag{B.5}$$

Plugging (B.5) into the final equality of (B.4) the series of the terms $i\mu_i$ will canceled, and we obtain

$$\begin{aligned}\mathbb{E}_{\hat{\mu}}(F) &= -\frac{G\beta_-}{\gamma}\left(1 + \mu_1\frac{\gamma_-}{\gamma_+}\right) - \frac{\beta_+}{\gamma}\sum_{i=2}^{\infty}\mu_i\left(q^{i-1} - \frac{1-q^i}{p}\right) \\ &= \frac{1}{\gamma}\left(\frac{\beta_+}{p} - G\beta_-\right) - \frac{\mu_1}{\gamma}\left(\frac{G\beta_-\gamma_-}{\gamma_+} + \frac{\beta_+}{p}\right) + \frac{\beta_+}{\gamma pq}\sum_{i=2}^{\infty}\mu_i q^i \\ &= \frac{1}{\gamma}\left(\frac{\beta_+}{p} - G\beta_-\right) - \frac{\mu_1}{\gamma}\left(\frac{G\beta_-\gamma_-}{\gamma_+} + \frac{2\beta_+}{p}\right) + \frac{\beta_+}{\gamma pq}\sum_{i=1}^{\infty}\mu_i q^i\end{aligned}$$

Soon, using the relation $\mu_k = \lambda_k \gamma$ we rewrite the latter into the terms of distribution $\lambda = \{\lambda_k\}_{k=1}^{\infty}$, and then applying (14) we finally obtain

$$\begin{aligned}\mathbb{E}_{\hat{\mu}}(F) &= \frac{1}{\gamma}\left(\frac{\beta_+}{p} - G\beta_-\right) - \lambda_1\left(\frac{G\beta_-\gamma_-}{\gamma_+} + \frac{2\beta_+}{p}\right) + \frac{\beta_+}{pq}\Psi(q) \\ &= \frac{1}{\gamma}\left(\frac{\beta_+}{p} - G\beta_-\right) - \lambda_1\left(\frac{G\beta_-\gamma_-}{\gamma_+} + \frac{2\beta_+}{p} + \frac{\beta_+ q}{p\gamma_-}\left(\gamma + \frac{\gamma_-}{q}\right)\right) \\ &= \frac{1}{\gamma}\left(\frac{\beta_+}{p} - G\beta_-\right) - \lambda_1\left(\frac{G\beta_-\gamma_-}{\gamma_+} + \frac{\beta_+}{p}\left(3 + \frac{q\gamma}{p\gamma_-}\right)\right).\end{aligned}$$

□

In the proof of the Theorem 5.2 we need to prove that the variance of the drift is finite. While the precise calculation of this variance is not presented herein, we offer a pivotal formula that proves to be instrumental in establishing its finiteness.

$$\sum_{j=1}^{i-1} j^2 p(i,j) = \frac{\gamma_-}{\gamma}\left(q^{i-1} + \frac{(i-1)^2}{p^2} + \frac{q^2 i^2}{p^2} - \frac{q}{p^2}\left(2i^2 - 2i - 1\right) - \frac{q^i}{p}\right). \tag{B.6}$$

Appendix C. Foster's criterion

For this section, we incorporated certain results obtained from [27].

Theorem C.1 (Theorem 2.6.4 (Foster's criterion), [27])**.** *An irreducible Markov chain X_n on a countable state space Σ is positive recurrent if and only if there exists a positive function $f : \Sigma \to \mathbb{R}_+$, a finite non-empty set $A \subset \Sigma$, and $\varepsilon > 0$ such that*

$$\mathbb{E}\left(f(X_{n+1}) - f(X_n) \mid X_n = x\right) \le -\varepsilon, \text{ for all } x \in \Sigma \setminus A, \tag{C.7}$$
$$\mathbb{E}\left(f(X_{n+1}) \mid X_n = x\right) < \infty, \text{ for all } x \in A. \tag{C.8}$$

Theorem C.2 (Theorem 2.5.19 from [27].)**.** *Assume that, for an irreducible Markov chain X_n on a countable state space Σ, one can find a function $f : \Sigma \to \mathbb{R}_+$ and $a \in (0, \infty)$, such that the set $A := \{x \in \Sigma : f(x) < a\} \neq \emptyset$ is a proper subset of Σ, and for some $\varepsilon > 0, \delta > 0$, and $B \in \mathbb{R}_+$, such that*

$$\mathbb{E}\left(f(X_{n+1}) - f(X_n) \mid X_n = x\right) \ge \varepsilon, \text{ for all } x \in \Sigma \setminus A; \text{and} \tag{C.9}$$
$$\mathbb{E}\left(|f(X_{n+1}) - f(X_n)|^{1+\delta} \mid X_n = x\right) < B, \text{ for all } x \in \Sigma \setminus A. \tag{C.10}$$

Then, the Markov chain is transient.

Define the following conditions.

M0 Let X_n be an irreducible, time-homogeneous Markov chain on Σ, a locally finite, unbounded subset of $\mathbb{R}_+$, with $0 \in \Sigma$.
M1 $\sup_{x\in\Sigma} \mathbb{E}\left(|X_{n+1} - X_n|^p \mid X_n = x\right) < \infty$ holds for some $p > 2$.
M2 Suppose that there exist $a \in \mathbb{R}$ and $b \in (0, \infty)$ such that

$$\lim_{x\to\infty} \mathbb{E}\left(|X_{n+1} - X_n|^2 \mid X_n = x\right) = b \text{ and } \lim_{x\to\infty} \mathbb{E}\left(X_{n+1} - X_n \mid X_n = x\right) = a.$$

Consequence of Theorem 3.2.3, [27]. *If conditions **M0**, **M1** and **M2** all hold, then X_n is*

- *transient if $2a > b$;*
- *null recurrent if $|2a| < b$;*
- *positive recurrent if $2a < -b$.*

Appendix D. Auxiliary lemmas for large deviation result

Consider the following compound Poisson process

$$\hat{S}(t) := \sum_{k=1}^{\nu(t)} \hat{\zeta}_k,\ t \in \mathbb{R}_+,$$

where i.i.d. random variables $\hat{\zeta}_1, \dots, \hat{\zeta}_k, \dots$ don't depend on Poisson process $\nu(\cdot)$ with rate γ, and

$$\mathbb{P}(\hat{\zeta}_1 = G) = \frac{\gamma_+}{\gamma},\ \mathbb{P}(\hat{\zeta}_1 = -l) = \frac{\gamma_-}{\gamma} p q^{l-1},\ l \in \mathbb{N}.$$

As an auxiliary result, we will be interested in the $\mathbb{G}$-LLDP for the family of processes

$$\hat{S}_T(t) := \frac{S(Tt)}{T},\ t \in [0,1]. \tag{D.11}$$

Lemma D.1. *The family of the processes $\hat{S}_T(\cdot)$ satisfies an $\mathbb{G}$-LLDP with the normalizing function $\psi(T) = T$ and the rate function*

$$I(f) = \begin{cases} \int_0^1 \Lambda(f_a'(t))dt - (\ln q) f_s^-(1),\ f \in \mathbb{G} \cap \mathbb{A}, \\ \infty,\ f \in \mathbb{G} \setminus \mathbb{A}. \end{cases}$$

Proof. Let $(\mathbb{D}[0,1], \rho_B)$ be the space of càdlàg functions with Borovkov metric (see [28, 29] for the definition of the metric ρ_B). It's easy to prove the following properties:

1. $\rho_B(f,g) \le \rho(f,g)$ for any $f, g \in \mathbb{D}[0,1]$;
2. If $f \in \mathbb{C}[0,1]$, $g_v \in \mathbb{D}[0,1]$, $v \in \mathbb{R}_+$ and $\lim_{v\to\infty} \rho_B(f, g_v) = 0$ then $\lim_{v\to\infty} \rho(f, g_v) = 0$.

We note that the second property follows from the uniform continuity of the function $f(\cdot)$ on $[0,1]$. It follows from Theorem 1.1 of [28] and properties 1), 2) that it suffices to show that

$$\ln \mathbb{E} e^{\lambda \hat{S}(1)} = A(\lambda), \tag{D.12}$$

$$A(\lambda) < \infty,\ \text{for } \lambda \in (\ln q, \infty). \tag{D.13}$$

It is easy to see that condition (D.13) is satisfied, so we have to check the condition (D.12).

Using the Beppo Levi's lemma and the fact that i.i.d. random variables $\hat{\zeta}_1, \dots, \hat{\zeta}_k, \dots$ don't depend on $\nu(\cdot)$, we obtain

$$\begin{aligned} \mathbb{E} e^{\lambda \hat{S}(1)} &= \mathbb{E} \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k} e^{\lambda \hat{\zeta}_r} \right) \mathbf{I}(\nu(1) = k) \\ &= \sum_{k=0}^{\infty} \mathbb{E} \left(\prod_{r=0}^{k} e^{\lambda \hat{\zeta}_r} \right) \mathbf{I}(\nu(1) = k) = \sum_{k=0}^{\infty} \left(\prod_{r=0}^{k} \mathbb{E} e^{\lambda \hat{\zeta}_r} \right) \mathbb{E}\mathbf{I}(\nu(1) = k) \\ &= e^{-\gamma} \sum_{k=0}^{\infty} \left(\frac{\gamma_+}{\gamma} e^{\lambda G} + \frac{\gamma_- p}{\gamma} \cdot \frac{e^{-\lambda}}{1 - q e^{-\lambda}} \right)^k \frac{\gamma^k}{k!} \\ &= \exp\left\{ \gamma_+ e^{\lambda G} + \gamma_- p \frac{e^{-\lambda}}{1 - q e^{-\lambda}} - \gamma \right\} = e^{A(\lambda)}. \end{aligned}$$

□

Lemma D.2. *Let $u > 0$, and consider the set A belonging to the Borel σ-algebra generated by open cylinder sets in the space $(\mathbb{D}[0,1],\rho)$, and let*

$$\{S. \in A\} \subseteq \left\{\min_{t\geq u} S_t \geq 2\right\}, \quad \{\hat{S}. \in A\} \subseteq \left\{\min_{t\geq u} \hat{S}_t \geq 2\right\}.$$

Then, for any $m_0 \geq 1$ the following equality holds true

$$\mathbb{P}(S. \in A \mid S_u = m_0) = \mathbb{P}(\hat{S}. \in A \mid \hat{S}_u = m_0).$$

Proof. Because the time instances of jumps in the processes S and $\hat{S}$ coincide, and the size of the jumps does not depend on the occurrence time, we only need to demonstrate that the following equality holds for any $m_k \in \mathbb{Z} \cap [2,\infty)$, where $k \in \mathbb{Z}_+$.

$$\begin{aligned}&\mathbb{P}(S_{t_1} = m_1, \ldots, S_{t_k} = m_k \mid S_u = m_0)\\ &= \mathbb{P}(\hat{S}_{t_1} = m_1, \ldots, \hat{S}_{t_k} = m_k \mid \hat{S}_u = m_0).\end{aligned}$$

where t_k is k-th jump of the process $\nu(t) - \nu(u)$, $t \geq u$.

The proof will be carried out by the method of mathematical induction. Let $l = 1$, then

$$\begin{aligned}&\mathbb{P}(S_{t_1} = m_1 \mid S_u = m_0) = \mathbb{P}(S_{t_1} - S_u = m_1 - m_0 \mid S_u = m_0)\\ &= \begin{cases} \dfrac{\gamma_+}{\gamma}, \text{ if } m_1 - m_0 = 1;\\ \dfrac{\gamma_-}{\gamma} pq^{m_0-m_1-1}, \text{ if } 2 \leq m_1 \leq m_0 - 1;\end{cases}\\ &= \mathbb{P}(\hat{S}_{t_1} - \hat{S}_u = m_1 - m_0 \mid \hat{S}_u = m_0) = \mathbb{P}(\hat{S}_{t_1} = m_1 \mid \hat{S}_u = m_0).\end{aligned} \tag{D.14}$$

Note that the equality (D.14) will maintain when in condition instead of $S_u = m_0$, $\hat{S}_u = m_0$ we will write $S_{t_{-1}} = m_0$, $\hat{S}_{t_{-1}} = m_0$ correspondingly, if we set t_{-1} for the last jump of the process $\nu(\cdot)$ before the time instant u.

Let for $l = k-1$

$$\begin{aligned}&\mathbb{P}(S_{t_1} = m_1, \ldots, S_{t_l} = m_l \mid S_u = m_0)\\ &= \mathbb{P}(\hat{S}_{t_1} = m_1, \ldots, \hat{S}_{t_l} = m_l \mid \hat{S}_u = m_0).\end{aligned} \tag{D.15}$$

We will show that the equality (D.15) holds for $l = k$. Using the inductive assumption, equality (2), and Markov property, we obtain

$$\begin{aligned}&\mathbb{P}(S_{t_1} = m_1, \ldots, S_{t_k} = m_k \mid S_u = m_0)\\ &= \mathbb{P}(S_{t_k} = m_k \mid S_u = m_0, S_{t_1} = m_1, \ldots, S_{t_{k-1}} = m_{k-1})\\ &\times \mathbb{P}(S_{t_1} = m_1, \ldots, S_{t_k} = m_{k-1} \mid S_u = m_0)\\ &= \mathbb{P}(S_{t_k} = m_k \mid S_{t_{k-1}} = m_{k-1})\mathbb{P}(\hat{S}_{t_1} = m_1, \ldots, \hat{S}_{t_k} = m_{k-1} \mid S(u) = m_0)\\ &= \mathbb{P}(\hat{S}_{t_k} = m_k \mid \hat{S}_{t_{k-1}} = m_{k-1})\mathbb{P}(\hat{S}_{t_1} = m_1, \ldots, \hat{S}_{t_k} = m_{k-1} \mid S(u) = m_0)\\ &= \mathbb{P}(\hat{S}_{t_1} = m_1, \ldots, \hat{S}_{t_k} = m_k \mid \hat{S}_u = m_0).\end{aligned}$$

□

Lemma D.3. *Let*

$$\Lambda(y) = \sup_{\lambda \in \mathbb{R}} (\lambda y - A(\lambda)), \ \ y \in \mathbb{R},$$

then

$$\Lambda_\delta(y) = \sup_{\lambda \in \mathbb{R}} (\lambda y - (1-\delta) A(\lambda)) = (1-\delta)\Lambda\left(\frac{y}{(1-\delta)}\right).$$

Proof. It's easy to see that

$$\begin{aligned} \sup_{\lambda \in \mathbb{R}} (\lambda y - (1-\delta) A(\lambda)) &= (1-\delta) \sup_{\lambda \in \mathbb{R}} \left(\lambda \frac{y}{(1-\delta)} - A(\lambda)\right) \\ &= (1-\delta)\Lambda\left(\frac{y}{(1-\delta)}\right). \end{aligned}$$

□

Lemma D.4. *Let $f \in \mathbb{A} \cap \mathbb{G}$. There is a constant $M > 0$ such that for any $\delta \in (0,1)$ and sufficiently small $\varepsilon > 0$ the inequality*

$$\liminf_{T \to 0} \frac{1}{T} \ln \mathbb{P}\left(\bigcap_{l=1}^{l_{\delta,\varepsilon}} A_l\right) \geq -4M \max(\, Varf_{[0,\delta]}, \delta)$$

holds.

Proof. Let $f(u_l) - f(u_{l-1}) > 0$. In this case for $f \in \mathbb{G} \cap \mathbb{A}$

$$f(u_l) - f(u_{l-1}) \leq f_a(u_l) - f_a(u_{l-1}). \tag{D.16}$$

Using the Lipschitz property of the function $f_a(\cdot)$ and (D.16), we obtain for some constant L and sufficiently large T

$$\begin{aligned} &\mathbb{P}\left(A_l \,|\, S_{Tu_{l-1}} = \lfloor Tf(u_{l-1}) \rfloor\right) \\ &= \mathbb{P}(B_{l,1}, B_{l,2}) = e^{-\gamma T(u_l - u_{l-1})} \frac{(\gamma_+ T(u_l - u_{l-1}))^{k_l}}{k_l!} \\ &\geq \exp\left\{-\gamma T(u_l - u_{l-1}) + k_l \ln\left(\frac{\gamma_+ T(u_l - u_{l-1})}{k_l}\right)\right\} \\ &\geq \exp\left\{-\gamma T(u_l - u_{l-1}) + k_l \ln\left(\frac{\gamma_+ T(u_l - u_{l-1})}{LT(u_l - u_{l-1})}\right)\right\} \\ &\geq \exp\left\{-\gamma T(u_l - u_{l-1}) - k_l \left|\ln\left(\frac{\gamma_+}{L}\right)\right|\right\} \\ &\geq \exp\left\{-T\left(\gamma(u_l - u_{l-1}) - \frac{2}{G}(f_a(u_l) - f_a(u_{l-1}))\left|\ln\left(\frac{\gamma_+}{L}\right)\right|\right)\right\} \\ &\geq \exp\left\{-T\left(\gamma + \frac{2L}{G}\left|\ln\left(\frac{\gamma_+}{L}\right)\right|\right)(u_l - u_{l-1})\right\}, \end{aligned} \tag{D.17}$$

here $k_l = \lfloor \frac{Tf(u_l) - Tf(u_{l-1})}{G} \rfloor$.

Let $f(u_l) - f(u_{l-1}) < 0$ then we have for sufficiently large T

$$
\begin{aligned}
&\mathbb{P}\left(A_l \,|\, S_{Tu_{l-1}} = \lfloor Tf(u_{l-1})\rfloor\right) = \mathbb{P}(C_{l,1}, C_{l,2} \,|\, S_{Tu_{l-1}} = \lfloor Tf(u_{l-1})\rfloor) \\
&= e^{-\gamma T(u_l - u_{l-1})}\gamma_- T(u_l - u_{l-1}) p q^{|k_l|-1} \\
&\geq \exp\left\{-\gamma T(u_l - u_{l-1}) + \ln(\gamma_- T(u_l - u_{l-1})p) - |k_l \ln q|\right\} \\
&\geq \exp\left\{-2T\max(|f(u_l) - f(u_{l-1})|, u_l - u_{l-1})(\gamma + |\ln q|)\right\}.
\end{aligned}
\tag{D.18}
$$

Let's

$$
M := \gamma + |\ln q| + \frac{L}{G}\left|\ln\left(\frac{\gamma_+}{L}\right)\right|.
$$

Using last expressions in inequalities (D.17), (D.18) and Markov property we have

$$
\begin{aligned}
\mathbb{P}\left(\bigcap_{l=1}^{l_{\delta,\varepsilon}} A_l\right) &\geq \exp\left\{-2TM\sum_{l=1}^{l_\delta}\max(|f(u_l) - f(u_{l-1})|, u_l - u_{l-1})\right\} \\
&\geq \exp\left\{-4TM\max(\mathrm{Var} f_{[0,\delta]}, \delta)\right\}.
\end{aligned}
\tag{D.19}
$$

It follows from the inequality (D.19) that

$$
\liminf_{T\to 0}\frac{1}{T}\ln\mathbb{P}\left(\bigcap_{l=1}^{l_{\delta,\varepsilon}} A_l\right) \geq -4M\max(\mathrm{Var} f_{[0,\delta]}, \delta).
$$

□

Acknowledgements

AY thanks FAPESP for financial support under the grant 2017/10555-0. JCH thanks the Unidad de Investigación, FIEECS-UNI and PROCIENCIA, for financial support. Some of this work was conducted during Professor Logachov's visit to UNI, under grant 427-2019 from PROCIENCIA. AL is also supported by the Ministry of Science and Higher Education of the Russian Federation FWNF-2022-0010.

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